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PERTURBATION THEORY FOR ARBITRATION GAMES

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ABSTRACT

In this paper, the effect on values and ε-optimal strategies of perturbations of the game parameters, such as payoff functions and arbitration functions, is studied for arbitration games. One main result is that the value is an upper semicontinuous function, if the underlying bargaining solution is upper semicontinuous. Furthermore, it appears that optimal behaviour for the players in an arbitration game is almost optimal if the game is slightly perturbed.

1. INTRODUCTION

Arbitration games correspond, roughly speaking, to situations where two players of a game in normal form are willing to cooperate and where they submit the problem how to cooperate to an arbitrator. The role of each of the players is, in this new situation, to deliver a threat strategy to the arbitrator. Based on the payoff pair corresponding to this pair of strategies the arbitrator determines, with the help of a known bargaining solution, a payoff point agreeable for both players.

Such games were introduced independently of each other by NASH [8] and RAIFFA [9]. Many authors contributed to the problem of the existence of value and optimal threat strategies for those games. In TIJS and JANSEN [10] a string of existence results was obtained, containing all known existence results. Also in Section 5 of this paper we will prove an existence result for convex continuous games.

However, the main purpose of this paper is to find an answer to the question how value and ε-optimal threat strategies depend on the payoff functions and the underlying bargaining solution of an arbitration game. This question is not only of theoretical importance, but favorable answers to it will give greater confidence in the application of the model in practice. We will find favorable answers in Theorem 4 of Section 4, where we show that optimal behaviour for the players in an arbitration game is almost optimal if the game is slightly perturbed. Notice, that in practice the payoff functions of a game to be played are only given with a certain (amount of) precision.

To obtain the before mentioned result we first study the dependence of the value on the payoff functions. It appears that the value depends on the game parameters in an upper semicontinuous way if an upper semicontinuous bargaining solution is used. Because this dependence is not a continuous one, the perturbation theory for arbitration games becomes more technical than for other classes of games (cf. TIJS and VRIEZE [11]). This lack of continuity is caused by the fact that there do not exist continuous bargaining solutions (cf. JANSEN and TIJS [4]).

The organization of the paper is as follows. The necessary facts about bargaining solutions and arbitration games are collected in the Sections 2 and 3. We study in the Sections 4 and 5 the effects of perturbing the payoff functions and in Section 6 the consequences of perturbation of the bargaining solution are considered.

2. BARGAINING SOLUTIONS

Pairs (a,S), where $S \subset \mathbb{R}^2$ is compact and convex and $a \in S$ are called *bargaining pairs*. The set of all bargaining pairs is denoted by $\mathcal{B}$. An element $(a,S) \in \mathcal{B}$ corresponds, intuitively, to a situation, where two players are involved and where the i-th coordinate a_i of a is the level of utility that player i receives, if they do not cooperate with each other, while S contains all the attainable points when they cooperate. Of course, for $s \in S$, s_i is the utility of outcome s for player $i \in \{1,2\}$.

In the following, the family of non-empty compact subsets of $\mathbb{R}^2$ is denoted by $\mathcal{C}$ and the family of convex sets in $\mathcal{C}$ by $\mathcal{K}$. We suppose that $\mathcal{C}$ is provided with the Hausdorff metric $d_H\colon \mathcal{C} \times \mathcal{C} \to \mathbb{R}$, defined by

$$d_H(S,T) = \inf\{\varepsilon > 0;\ S \subset B_\varepsilon(T),\ T \subset B_\varepsilon(S)\}, \quad ((S,T) \in \mathcal{C})$$

where $B_\varepsilon(S) = \{x \in \mathbb{R}^2;\ \inf_{s\in S} \|x-s\|_\infty \le \varepsilon\}$. [$\|\cdot\|_\infty$ is the usual maximum norm in $\mathbb{R}^2$.]

We provide $\mathcal{B}$ with the metric $d\colon \mathcal{B}\times\mathcal{B} \to \mathbb{R}$, defined by

$$d((a,S),(b,T)) = \max\{\|a-b\|_\infty,\ d_H(S,T)\},$$
$$\text{for all } (a,S),(b,T) \in \mathcal{B}.$$

For each $S \in K$, the *Pareto set* of S

$$\{p \in S;\ \text{for each } s \in S \text{ with } s \ge p, \text{ we have } s = p\}$$

is denoted by $P(S)$ and the *weak Pareto set* of S

$$\{w \in S;\ \text{for each } s \in S \text{ with } s \ge w, \text{ we have } s_1 = w_1 \text{ or } s_2 = w_2\}$$

is denoted by $W(S)$.

In [4], we proved that, for a sequence $S_1, S_2, \ldots$ in K converging to $S \in K$, the following holds:

(2.1) $$\limsup_{n\to\infty} W(S_n) \subset W(S)$$

(2.2) $$\liminf_{n\to\infty} P(S_n) \supset P(S).$$

Here $\liminf_{n\to\infty} S_n$ consists of those points $x \in \mathbb{R}^2$, for which there exists a sequence $s(1), s(2), \ldots$ such that $s(n) \in S_n$, for each $n \in \mathbb{N}$, and $\lim_{n\to\infty} s(n) = x$; $\limsup_{n\to\infty} S_n$ consists of those points $x \in \mathbb{R}^2$, for which there are a subsequence $n(1), n(2), \ldots$ of $1, 2, \ldots$ and $s(n(1))$, $s(n(2)), \ldots$ such that $s(n(k)) \in S_{n(k)}$, for each $k \in \mathbb{N}$, and $\lim_{k\to\infty} s(n(k)) = x$.

A map $\phi\colon \mathcal{B} \to \mathbb{R}^2$ is called a *bargaining solution* if

(B.1) $\phi(a,S) \in P(S)$, for each $(a,S) \in \mathcal{B}$ *(Pareto optimality)*

(B.2) $\phi(a,S) \ge a$, for each $(a,S) \in \mathcal{B}$ *(Individual rationality)*

A bargaining solution is called *upper semicontinuous* (u.s.c.) if

(B.3) for each sequence $(a,S), (a(1),S_1), (a(2),S_2), \ldots$ in $\mathcal{B}$ with

$$\lim_{n\to\infty} (a(n),S_n) = (a,S), \quad \text{we have, for } i \in \{1,2\},$$

$$\phi_i(a,S) \geq \limsup_{n\to\infty} \phi_i(a(n),S_n).$$

In [4], it was proved that there do not exist continuous bargaining solutions and that the solutions proposed by NASH [7], KALAI and ROSENTHAL [5], KALAI and SMORODINSKY [6] and YU [12], are all u.s.c.. A bargaining solution $\phi\colon \mathcal{B} \to \mathbb{R}^2$ is called *regular* if
(B.4) for each $(a,S),(b,S) \in \mathcal{B}$ with $\phi(a,S) = \phi(b,S)$, we have

$$\phi(\lambda a+(1-\lambda)b,S) = \phi(a,S), \quad \text{for all } \lambda \in [0,1].$$

In the following, a role is played by the u.s.c. and regular bargaining solutions ψ^1 and ψ^2, defined as follows. For $(a,S) \in \mathcal{B}$ and $i \in \{1,2\}$, $\psi^i(a,S)$ is the element in $\{p \in P(S);\ p \geq a\}$ with maximal i-th coordinate.

We introduce here some notation. For an element $S \in K$, we define

$$\bar{p}(S) = \Big(\min_{p\in P(S)} p_1, \max_{p\in P(S)} p_2\Big), \underline{p}(S) = \Big(\max_{p\in P(S)} p_1, \min_{p\in P(S)} p_2\Big)$$

The following lemmas were proved in [4].

<u>LEMMA 2.1</u>. (Cf. [4], pp. 12,13). *Let* $\phi\colon \mathcal{B} \to \mathbb{R}^2$ *be an u.s.c. bargaining solution. Let* $(a,S) \in \mathcal{B}$. *Then we have:*
(i) *if* $\phi(a,S) \neq \underline{p}(S)$, *then* ϕ_2 *is continuous in* (a,S);
(ii) *if* $\phi(a,S) \neq \bar{p}(S)$, *then* ϕ_1 *is continuous in* (a,S).

[$\phi_i(a,S)$ is the i-th coordinate of $\phi(a,S)$.]

<u>LEMMA 2.2</u>. (Cf. [4], p.10). *Let* $\phi\colon \mathcal{B} \to \mathbb{R}^2$ *be an u.s.c. bargaining solution. Let* $S \in K$ *and let* $a,a(1),a(2),\ldots$ *be a sequence in* S, *such that* $\lim_{n\to\infty} a(n) = a$. *Then* $\lim_{n\to\infty} \phi(a(n),S) = (a,S)$.

For later use, we mention still two lemmas. The proofs of these lemmas are left to the reader.

<u>LEMMA 2.3</u>. *For each* $S,T \in C$, $\text{conv}(S),\text{conv}(T) \in K$ *and*

$$d_H(\mathrm{conv}(S),\mathrm{conv}(T)) \le d_H(S,T).$$

[conv(S) denotes the convex hull of S.]

LEMMA 2.4. *For each bounded set* V *in* $\mathbb{R}^2$, $\mathrm{conv}(\mathrm{cl}(V)) = \mathrm{cl}(\mathrm{conv}(V))$, *where* cl(V) *is the closure of* V.

3. ARBITRATION GAMES

We consider two-person games in normal form $\Gamma = \langle X,Y,K_1,K_2\rangle$, where X is a non-empty set (the *strategy space of player* 1), Y is a non-empty set (the *strategy space of player* 2), and $K_i : X\times Y \to \mathbb{R}$ is, for $i \in \{1,2\}$, a bounded real-values function on the Cartesian product of the strategy spaces (the *payoff function* for player i). We suppose that the players may use *correlated strategies* i.e. discrete probability measures on $X \times Y$. The set

$$\mathcal{R}_0(\Gamma) = \{K(x,y) = (K_1(x,y),K_2(x,y)) \in \mathbb{R}^2;\ \ (x,y) \in X\times Y\},$$

is the set of attainable payoff pairs, if there is no cooperation.

The closure of the convex hull of $\mathcal{R}_0(\Gamma)$ is denoted by $\mathcal{R}(\Gamma)$ and is called the *cooperative payoff space*, e.g. $\mathcal{R}(\Gamma) = \mathrm{cl}(\mathrm{conv}(\mathcal{R}_0(\Gamma)))$. By using correlated strategies, each point of $\mathcal{R}(\Gamma)$ can be approached as near as the players want. The Pareto set and the weak Pareto set of the non-empty compact and convex set $\mathcal{R}(\Gamma)$ will be denoted by $\mathcal{P}(\Gamma)$ and $\mathcal{W}(\Gamma)$, respectively. A function $\phi: \mathcal{R}(\Gamma) \to \mathcal{P}(\Gamma)$ is called an *arbitration function* if

(A.1) $\phi(r) \ge r$, for each $r \in \mathcal{R}(\Gamma)$;

(A.2) ϕ is continuous.

An arbitration function ϕ is called *regular* if

(A.3) for all $r,s \in \mathcal{R}(\Gamma)$ with $\phi(r) = \phi(s)$, we have

$$\phi(\lambda r + (1-\lambda)s) = \phi(r), \quad \text{for all } \lambda \in (0,1).$$

Suppose that the players of Γ decide to call in an arbitrator, who helps the players in the choice of an element of $\mathcal{P}(\Gamma)$, with the aid of an arbitration function $\phi: \mathcal{R}(\Gamma) \to \mathcal{P}(\Gamma)$. The situation then proceeds as follows:

Step 1. Independently of each other, the players assign an $x^0 \in X$ and a $y^0 \in Y$ and deliver it to the arbitrator.

Step 2. The arbitrator calculates the payoff $\phi(K_1(x^0,y^0),K_2(x^0,y^0))$ and chooses a correlated strategy μ, such that the expected payoff $\iint K(x,y)d\mu(x,y)$ with respect to μ equals $\phi(K(x^0,y^0))$, if that is possible; otherwise μ is chosen, such that $\iint K(x,y)d\mu(x,y)$ is as close to $\phi(K(x^0,y^0))$ as both players want.

Step 3. With a lottery corresponding to μ, an outcome $(x^1,y^1) \in X \times Y$ is determined.

Step 4. The non-cooperative game Γ is played, where players 1 and 2 are obliged to choose x^1 and y^1, respectively, resulting in a payoff $K_i(x^1,y^1)$ for player $i \in \{1,2\}$.

From a strategic point of view, for the players, this new game is, essentially, the non-cooperative game in normal form $\Gamma_\phi = \langle X,Y,\phi_1K,\phi_2K\rangle$, where $\phi_iK(x,y)$ is the i-th coordinate of $\phi(K_1(x,y),K_2(x,y)) \in \mathbb{R}^2$. This game Γ_ϕ is called the *arbitration game,* corresponding to the game Γ and the arbitration function ϕ. Such an arbitration game Γ_ϕ has many similarities with zero-sum games, because also for an arbitration game the preference relations of the players on the outcome set $\{(\phi_1K(x,y), \phi_2K(x,y));\ (x,y) \in X \times Y\}$ are strictly opposite. For Γ_ϕ, the expressions $v_1(\Gamma_\phi) = \sup_{x\in X} \inf_{y\in Y} \phi_1K(x,y)$ and $v_2(\Gamma_\phi) = \sup_{y\in Y} \inf_{x\in X} \phi_2K(x,y)$ are called the *security levels* for player 1 and player 2, respectively.

DEFINITION 3.1. Let Γ_ϕ be an arbitration game and let $\varepsilon \geq 0$. We say that Γ_ϕ is *strictly determined,* if $(v_1(\Gamma_\phi), v_2(\Gamma_\phi)) \in \mathcal{P}(\Gamma)$. In that case, $v_i(\Gamma_\phi)$ is called the *(arbitration) value* for player i and $v(\Gamma_\phi) = (v_1(\Gamma_\phi), v_2(\Gamma_\phi))$ the *value* of the arbitration game. If Γ_ϕ is strictly determined, then $x^* \in X$ is called an ε-*optimal strategy* for player 1, if $\phi_1K(x^*,y) \geq v_1(\Gamma_\phi) - \varepsilon$, for all $y \in Y$. ε-Optimal strategies for player 2 can be defined, similarly. 0-Optimal strategies are also called optimal strategies.

We note, that a pair of strategies $(x^*,y^*) \in X \times Y$ is a (Nash) equilibrium point of a strictly determined game Γ_ϕ iff x^* and y^* are optimal strategies for player 1 and player 2, respectively. In the following, the set of ε-optimal strategies of player i of a strictly determined game will be denoted by $O_i^\varepsilon(\Gamma_\phi)$ and the set of optimal strategies by $O_i(\Gamma_\phi)$, while the set of equilibrium points is denoted by $E(\Gamma_\phi)$.

In [10], many sufficient conditions are given for the existence of the value of an arbitration game. Also in Section 5, we will derive an existence theorem.

We need in the next sections the following notation. Let $\Gamma = \langle X,Y,K_1,K_2\rangle$ be a two-person game in normal form. Then

$$\bar{p}(\Gamma) = (\min_{p \in P(\Gamma)} p_1, \max_{p \in P(\Gamma)} p_2),$$

$$\underline{p}(\Gamma) = (\max_{p \in P(\Gamma)} p_1, \min_{p \in P(\Gamma)} p_2)$$

$j_\Gamma: W(\Gamma) \to P(\Gamma)$ is the map, defined by

$$j_\Gamma(w) = \begin{cases} \bar{p}(\Gamma) & w \in \bar{W}(\Gamma) \\ w & \text{if } w \in P(\Gamma) \\ \underline{p}(\Gamma) & w \in \underline{W}(\Gamma). \end{cases}$$

We conclude this section by relating bargaining solutions and arbitration functions. Let $\phi: B \to \mathbb{R}^2$ be an u.s.c. bargaining solution. Let $\Gamma = \langle X,Y,K_1,K_2\rangle$ be a two-person game. Then ϕ induces an arbitration function $\phi^\Gamma: R(\Gamma) \to P(\Gamma)$, where

$$\phi^\Gamma(r) = \phi(r,R(\Gamma)), \qquad \text{for all } r \in R(\Gamma).$$

Note that ϕ^Γ is the restriction of ϕ to the subset of bargaining pairs $B_\Gamma = \{(r,S);\ r \in R(\Gamma),\ S = R(\Gamma)\}$. That ϕ^Γ is continuous follows from Lemma 2.2. Instead of ϕ^Γ, we will often write ϕ in the following. Immediate is the following result.

LEMMA 3.2. *Let $\phi: B \to \mathbb{R}^2$ be an u.s.c. and regular bargaining solution and let Γ be a two-person game. Then ϕ^Γ is a regular arbitration function.*

4. PERTURBATIONS OF THE PAYOFF FUNCTIONS

In this section, X and Y are fixed non-empty sets. Let G be the set, consisting of two-person games in normal form $\langle X,Y,K_1,K_2\rangle$, where K_1 and K_2 are bounded payoff functions. Let $d\colon G \times G \to \mathbb{R}$ be the matric on G, defined by

$$d(\Gamma,\Gamma') = \sup_{(x,y)\in X\times Y} \|K(x,y) - K'(x,y)\|_\infty,$$

for all $\Gamma = \langle X,Y,K_1,K_2\rangle$, $\Gamma' = \langle X,Y,K_1',K_2'\rangle \in G$.

Let $\phi\colon B \to \mathbb{R}^2$ be an u.s.c. bargaining solution. Then $VG(\phi)$ denotes the set of games $\Gamma \in G$, such that the arbitration game Γ_ϕ possesses a value. In this section, we are mainly interested in the continuity properties of the map $\Gamma \to v(\Gamma_\phi)$ from $VG(\phi)$ into $\mathbb{R}^2$. We need a

<u>LEMMA 4.1</u>. *Let* $\Gamma = \langle X,Y,K_1,K_2\rangle$, $\Gamma' = \langle X,Y,K_1',K_2'\rangle$, $\Gamma^1,\Gamma^2,\ldots$ *be elements of* G. *Then*

(i) $d_H(R(\Gamma),R(\Gamma')) \le d(\Gamma,\Gamma')$.

(ii) *If* $\lim_{n\to\infty} d(\Gamma^n,\Gamma) = 0$, *then* $\limsup_{n\to\infty} W(\Gamma^n) \subset W(\Gamma)$.

(iii) *If* $\lim_{n\to\infty} d(\Gamma^n,\Gamma) = 0$, *then* $\liminf_{n\to\infty} P(\Gamma^n) \supset P(\Gamma)$.

<u>PROOF</u>. Take $r \in \mathrm{cl}(R_0(\Gamma))$. For each $\delta > 0$, there is an $(x,y) \in X\times Y$, such that $\|K(x,y) - r\|_\infty \le \delta$. Then

$$\|K'(x,y) - r\|_\infty \le \|K'(x,y) - K(x,y)\|_\infty + \|K(x,y) - r\|_\infty$$

$$\le \|K' - K\|_\infty + \delta.$$

Hence, $r \in B_{\delta+\|K'-K\|_\infty}(R_0(\Gamma')) \subset B_{\delta+\|K'-K\|_\infty}(\mathrm{cl}(R_0(\Gamma')))$ for each $\delta > 0$, which implies that $\mathrm{cl}(R_0(\Gamma)) \subset B_{\|K'-K\|_\infty}(\mathrm{cl}\ R_0(\Gamma'))$. Similarly, it follows that $\mathrm{cl}(R_0(\Gamma')) \subset B_{\|K'-K\|_\infty}(\mathrm{cl}(R_0(\Gamma)))$. So $d_H(\mathrm{cl}(R_0(\Gamma)), \mathrm{cl}(R_0(\Gamma'))) \le \|K'-K\|_\infty = d(\Gamma,\Gamma')$. Since, by Lemma 2.4,

$$R(\Gamma) = \mathrm{conv}(\mathrm{cl}(R_0(\Gamma))) \quad \text{and} \quad R(\Gamma') = \mathrm{conv}(\mathrm{cl}(R_0(\Gamma'))),$$

it follows from Lemma 2.3 that (i) holds.

To prove (ii) and (iii), we note that $\lim_{n\to\infty} d(\Gamma^n,\Gamma) = 0$ implies in view of (i), that $\lim_{n\to\infty} d_H(R(\Gamma^n),R(\Gamma)) = 0$. Then, by (2.1) and (2.2), the assertions (ii) and (iii) are true. □

First, we give an answer to the question, whether there exist bargaining solutions such that the corresponding value function is continuous.

PROPOSITION 4.2. *There exist no solutions* $\phi\colon B \to \mathbb{R}^2$, *such that* $\Gamma \to v(\Gamma_\phi)$ $(\Gamma \in VG(\phi))$ *is continuous.*

PROOF. Let Γ be the mixed extension of the bimatrix game (A,B) with $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. For $n \in \mathbb{N}$, let Γ^n be the mixed extension of the bimatrix game (A_n,B) with

$$A_n = \begin{bmatrix} 0 & 1 \\ 1+n^{-1} & 1+n^{-1} \end{bmatrix}.$$

Then, for all $\phi\colon B \to \mathbb{R}^2$, we have $v(\Gamma_\phi) = (1,1)$, because $P(\Gamma) = \{(1,1)\}$; and $v(\Gamma^n_\phi) = (1+n^{-1},0)$, for all $n \in \mathbb{N}$ (the second pure strategy of player 1 is optimal in Γ^n_ϕ). Now $\lim_{n\to\infty} d(\Gamma^n,\Gamma) = 0$ and $\lim_{n\to\infty} v(\Gamma^n_\phi) \neq v(\Gamma_\phi)$. Hence, the value function v is not continuous. □

One of the main results in this paper is obtained in the following theorem, namely, that the value functions v_1 and v_2 are at least upper semi-continuous.

THEOREM 4.3. *Let* X, Y *be non-empty sets and let* $\phi\colon B \to \mathbb{R}^2$ *be an u.s.c. bargaining solution. Then*

(i) *The map* $\Gamma \to v_1(\Gamma_\phi)$ $(\Gamma \in VG(\phi))$ *is u.s.c.* $(i = 1,2)$.

(ii) *Let*

$$VG(\phi)^* = \{\Gamma \in VG(\phi);\ v(\Gamma_\phi) \neq \underline{p}(\Gamma)\},$$

$$VG(\phi)_* = \{\Gamma \in VG(\phi);\ v(\Gamma_\phi) \neq \bar{p}(\Gamma)\}.$$

Then $VG(\phi)^*$ *and* $VG(\phi)_*$ *are open subsets of* $VG(\phi)$.

(iii) $v_1\colon VG(\phi)_* \to \mathbb{R}$ *and* $v_2\colon VG(\phi)^* \to \mathbb{R}$ *are continuous functions.*

(iv) v *is continuous in* Γ, *if* $v(\Gamma_\phi) \notin \{\underline{p}(\Gamma),\bar{p}(\Gamma)\}$.

(v) *Let* $\Gamma,\Gamma^1,\Gamma^2,\ldots$ *be a sequence in* $VG(\phi)$, *such that*

$$\lim_{n\to\infty} d(\Gamma^n,\Gamma) = 0.$$

Then

$$v(\Gamma_\phi) = j_\Gamma(\lim_{n\to\infty} v(\Gamma^n_\phi)).$$

PROOF.

(a) First we prove that v_2 is lower semicontinuous in a point $\Gamma \in VG(\phi)^*$. Take $\varepsilon > 0$ such that $v_2(\Gamma_\phi) - \varepsilon > \underline{p}_2(\Gamma)$. Let

$$A = \{r \in R(\Gamma);\ \phi_2(r) \geq v_2(\Gamma_\phi) - \varepsilon\}.$$

Then A is a non-empty compact subset of $\mathbb{R}^2$, because ϕ is continuous on $R(\Gamma)$. This implies that

$$D = \{(a,S) \in B;\ a \in A,\ S = R(\Gamma)\}$$

is a compact subset of B. Let $f\colon B \to \mathbb{R}$ be the function, defined by $f(a,S) = \underline{p}_1(S) - \phi_1(a,S)$. Then f is lower semicontinuous and the restriction of f to the compact set D is a positive function. This implies that we can find a $\delta > 0$ such that the restriction of f to

$$D^\delta = \{(b,T) \in B;\ b \in B_\delta(A),\ T \in H_\delta(R(\Gamma))\}$$

is positive. Here

$$B_\delta(A) = \{x \in \mathbb{R}^2;\ \inf_{a\in A} \|x-a\| \leq \delta\}$$

and

$$H_\delta(A) = \{B \in C;\ d_H(B,A) \leq \delta\}.$$

Now, D^δ is a compact subset of B (cf. [3], p.17) and $\phi_2\colon D^\delta \to \mathbb{R}$ is continuous by Lemma 2.1, because $\phi_1(b,T) < \underline{p}_1(T)$ for all $(b,T) \in D^\delta$. Hence $\phi_2\colon D^\delta \to \mathbb{R}$ is a uniformly continuous function. So we can take a $\delta_1 \in (0,\delta]$, such that for all $(a,S),(b,T) \in D^\delta$ we have

(4.1) $$|\phi_2(a,S) - \phi_2(b,T)| < \varepsilon \quad \text{if } d((a,S),(b,T)) \le \delta_1.$$

Now, take a $\bar{y} \in Y$ such that

(4.2) $$\phi_2 K(x,\bar{y}) \ge v_2(\Gamma_\phi) - \varepsilon, \quad \text{for all } x \in X.$$

Then $\{K(x,\bar{y}) \subset \mathbb{R}^2;\ x \in X\} \subset A$. Let $\tilde{\Gamma} = \langle X,Y,\tilde{K}_1,\tilde{K}_2\rangle$ be a game in $\mathcal{VG}(\phi)$ with $d(\tilde{\Gamma},\Gamma) \le \delta_1$. Then the inequalities $\|\tilde{K}(x,\bar{y}) - K(x,\bar{y})\|_\infty \le \delta_1 \le \delta$ for all $x \in X$ and $d_H(\mathcal{R}(\Gamma),\mathcal{R}(\tilde{\Gamma})) \le \delta_1 \le \delta$ imply, that $(\tilde{K}(x,\bar{y}),\mathcal{R}(\tilde{\Gamma})) \in D^\delta$. In view of (4.1), we obtain

(4.3) $$|\phi_2(\tilde{K}(x,\bar{y})) - \phi_2(K(x,\bar{y}))| < \varepsilon, \quad \text{for all } x \in X.$$

Hence, by (4.3) and (4.2),

$$\phi_2\tilde{K}(x,\bar{y}) \ge \phi_2 K(x,\bar{y}) - \varepsilon \ge v_2(\Gamma_\phi) - 2\varepsilon.$$

So,

$$v_2(\tilde{\Gamma}_\phi) = \sup_{y\in Y}\inf_{x\in X} \phi_2\tilde{K}(x,y) \ge \inf_{x\in X} \phi_2\tilde{K}(x,\bar{y}) \ge v_2(\Gamma_\phi) - 2\varepsilon.$$

We have proved that, for all games $\tilde{\Gamma} \in \mathcal{VG}(\phi)$ with $d(\Gamma,\Gamma') \le \delta_1$, we have $v_2(\tilde{\Gamma}_\phi) \ge v_2(\Gamma_\phi) - 2\varepsilon$. Hence, v_2 is l.s.c. in Γ.

(b) If $\Gamma \in \mathcal{VG}(\phi)$ and $v(\Gamma_\phi) = \bar{p}(\Gamma)$, then v_2 is u.s.c. in Γ, because, for all $\tilde{\Gamma} \in \mathcal{VG}(\phi)$, we have

$$v_2(\tilde{\Gamma}_\phi) \le \bar{p}_2(\tilde{\Gamma}) \le \bar{p}_2(\Gamma) + d(\Gamma,\tilde{\Gamma}) = v_2(\Gamma_\phi) + d(\Gamma,\tilde{\Gamma}).$$

In view of part (a) of this proof, v_2 is continuous for all games Γ with $v(\Gamma_\phi) = \bar{p}(\Gamma)$.

(c) Similarly, as in (a) and (b), we can prove that v_1 is l.s.c. in points $\Gamma \in \mathcal{VG}(\phi)_*$ and continuous in points Γ with $v(\Gamma_\phi) = \underline{p}(\Gamma)$.

(d) Now, let Γ be a game with $v(\Gamma_\phi) \notin \{\underline{p}(\Gamma),\bar{p}(\Gamma)\}$. Let Γ^1,Γ^2, be a sequence in $\mathcal{VG}(\phi)$, converging to Γ. Suppose that $\langle v(\Gamma_\phi^{n(k)});\ k \in \mathbb{N}\rangle$ is a subsequence of $\langle v(\Gamma_\phi^n);\ n \in \mathbb{N}\rangle$, converging to $t \in \mathbb{R}^2$. Then

$v(\Gamma_\phi) \le \lim_{k\to\infty} v(\Gamma_\phi^{n(k)}) = t$, in view of (a) and (c). Furthermore, $t \in \mathcal{W}(\Gamma)$, by (2.1) and $v(\Gamma_\phi) \in P(\Gamma)$. But then $v(\Gamma_\phi) = t = \lim_{k\to\infty} v(\Gamma_\phi^{n(k)})$. This implies that $\lim_{n\to\infty} v(\Gamma_\phi^n) = v(\Gamma_\phi)$. So we have proved that v is continuous in Γ.

(e) Suppose that $v(\Gamma_\phi) = \bar{p}(\Gamma)$ and let $\Gamma^1, \Gamma^2, \ldots$ be a sequence in $VG(\Gamma)$, converging to Γ, and suppose that $\lim_{n\to\infty} v(\Gamma_\phi^n) = t$. Then $t \in \mathcal{W}(\Gamma)$, by (2.1), and $t_2 = v_2(\Gamma_\phi) = \bar{p}_2(\Gamma)$ by (b) of this proof. Hence, $t_1 \le \bar{p}_1(\Gamma)$. This implies that v_1 is u.s.c. in Γ. Similarly, it follows that v_2 is u.s.c. in points with $v(\Gamma_\phi) = \underline{p}(\Gamma)$.

(f) Combining the foregoing results implies that we have proved (i), (iii) and (iv). That also (ii) holds, follows from the fact that

$$VG(\phi)^* = \{\Gamma \in VG(\phi);\ v_1(\Gamma_\phi) - \underline{p}_1(\Gamma) < 0\}$$

$$VG(\phi)_* = \{\Gamma \in VG(\phi);\ v_2(\Gamma_\phi) - \bar{p}_2(\Gamma) < 0\}$$

and that $\Gamma \to v_1(\Gamma_\phi) - \underline{p}_1(\Gamma)$ and $\Gamma \to v_2(\Gamma_\phi) - \bar{p}_2(\Gamma)$ are u.s.c. functions.

(g) Now we prove (v). If $v(\Gamma_\phi) \notin \{\bar{p}(\Gamma), \underline{p}(\Gamma)\}$, then by (iv) $v(\Gamma_\phi) = \lim_{n\to\infty} v(\Gamma_\phi^n) = j_\Gamma(\lim_{n\to\infty} v(\Gamma_\phi^n))$, for a sequence $\Gamma^1, \Gamma^2, \ldots$ in $VG(\phi)$ converging to Γ. If $v(\Gamma_\phi) = \bar{p}(\Gamma)$, then by (iii), $\lim_{n\to\infty} v_2(\Gamma_\phi^n) = v_2(\Gamma_\phi) = \bar{p}_2(\Gamma)$. Hence, $\lim_{n\to\infty} v(\Gamma_\phi^n) \in \bar{\mathcal{W}}(\Gamma)$ and $v(\Gamma_\phi) = \bar{p}(\Gamma) = j_\Gamma(\lim_{n\to\infty} v(\Gamma_\phi^n))$. Similarly, $v(\Gamma_\phi) = j_\Gamma(\lim_{n\to\infty} v(\Gamma_\phi^n))$, if $v(\Gamma_\phi) = \underline{p}(\Gamma)$. □

In the next theorem, four conditions on a game Γ are given, each of them guaranteeing that for each u.s.c. bargaining solution ϕ for which the value $v(\Gamma_\phi)$ exists, the corresponding value function on $VG(\phi)$ is continuous in the point Γ. In Section 5, we prove that, for a certain subclass of games, these conditions are also exhaustive.

<u>THEOREM 4.4</u>. *Let* $\Gamma = \langle X,Y,K_1,K_2\rangle$ *be a two-person game and let* $s_1(\Gamma) = \sup_{x\in X} \inf_{y\in Y} K_1(x,y)$, $s_2(\Gamma) = \sup_{y\in Y} \inf_{x\in X} K_2(x,y)$. *Let* $\phi: B \to \mathbb{R}^2$ *be an u.s.c. bargaining solution and suppose that* $\Gamma \in VG(\phi)$. *Then* $v: VG(\phi) \to \mathbb{R}^2$ *is continuous in* Γ, *if one of the following four conditions holds.*

$$\text{(V.1)} \qquad \mathcal{P}(\Gamma) = \mathcal{W}(\Gamma),$$

$$\text{(V.1)} \qquad s_1(\Gamma) \geq \bar{p}_1(\Gamma), \quad s_2(\Gamma) \geq \underline{p}_2(\Gamma),$$

$$\text{(V.3)} \qquad s_1(\Gamma) \geq \bar{p}_1(\Gamma), \quad \underline{\mathcal{W}}(\Gamma) = \{\underline{p}(\Gamma)\},$$

$$\text{(V.4)} \qquad s_2(\Gamma) \geq \underline{p}_2(\Gamma), \quad \overline{\mathcal{W}}(\Gamma) = \{\bar{p}(\Gamma)\}.$$

<u>PROOF</u>. Let $\Gamma^1 = \langle X,Y,K_1^1,K_2^1\rangle$, $\Gamma^2 = \langle X,Y,K_1^2,K_2^2\rangle, \ldots$ be a sequence in $\mathcal{VG}(\phi)$, converging to Γ. Then the sequence $v(\Gamma_\phi^1), v(\Gamma_\phi^2), \ldots$ is bounded. To prove the theorem, it is sufficient to show that each convergent subsequence of this sequence, converges to $v(\Gamma_\phi)$, if one of the four conditions holds. Let $v(\Gamma^{n(1)}), v(\Gamma^{n(2)}), \ldots$ be such a subsequence, with limit t. By (2.1), $t \in \mathcal{W}(\Gamma)$ and by Theorem 4.3, $v(\Gamma_\phi) = f_\Gamma(\lim_{k\to\infty} v(\Gamma_\phi^{n(k)}))$. We are ready, if we can show that $t \in \mathcal{P}(\Gamma)$, if Γ satisfies one of the four conditions.

(a) If (V.1) holds, then $t \in \mathcal{W}(\Gamma) = \mathcal{P}(\Gamma)$.

(b) Suppose that $s_1(\Gamma) \geq \bar{p}_1(\Gamma)$. Since, for each $n \in \mathbb{N}$,

$$v_1(\Gamma_\phi^n) = \sup_{x\in X} \inf_{y\in Y} \phi_1(K^n(x,y)) \geq \sup_{x\in X} \inf_{y\in Y} K_1^n(x,y) = s_1(\Gamma^n)$$

we have

$$\text{(4.4)} \qquad \lim_{k\to\infty} v_1(\Gamma_\phi^{n(k)}) \geq \lim_{k\to\infty} s_1(\Gamma^{n(k)}) = s_1(\Gamma) \geq \bar{p}_1(\Gamma).$$

Similarly, it follows that

$$\text{(4.5)} \qquad \lim_{k\to\infty} v_2(\Gamma_\phi^{n(k)}) \geq \underline{p}_2(\Gamma), \quad \text{if } s_2(\Gamma) \geq \underline{p}_2(\Gamma).$$

(c) If (V.2) holds, then (4.4) and (4.5) imply that $(t_1,t_2) = (\bar{p}_1(\Gamma), \underline{p}_2(\Gamma))$. Hence, $t \in \mathcal{P}(\Gamma)$.

(d) If (V.3) holds, then (4.4) implies that $t \in \mathcal{P}(\Gamma) \cup \underline{\mathcal{W}}(\Gamma) = \mathcal{P}(\Gamma)$.

(e) If (V.4) holds, then (4.5) implies that $t \in \mathcal{P}(\Gamma) \cup \overline{\mathcal{W}}(\Gamma) = \mathcal{P}(\Gamma)$.

This completes the proof. □

Note that the games Γ, satisfying (V.1), form a dense subset of $\mathcal{VG}(\phi)$.

The following theorem states that an (almost) optimal strategy in a disturbed game is almost optimal in the original game, if the perturbation is small.

THEOREM 4.5. *Let* $\phi\colon \mathcal{B} \to \mathbb{R}^2$ *be an u.s.c. bargaining solution. Let* $\Gamma \in VG(\phi)$ *and let* $\varepsilon \geq 0$, $\varepsilon' > 0$. *Then there exists a* $\delta > 0$ *such that, for* $i \in \{1,2\}$, $O_i^{\varepsilon}(\Gamma'_\phi) \subset O_i^{\varepsilon+\varepsilon'}(\Gamma_\phi)$, *for all* $\Gamma' \in VG(\phi)$ *with* $d(\Gamma,\Gamma') < \delta$.

PROOF. We only prove the assertion for $i = 2$. If $\varepsilon \geq v_2(\Gamma_\phi) - \underline{p}_2(\Gamma)$, then

$$O_2^{\varepsilon}(\Gamma'_\phi) \subset Y = O_2^{\varepsilon+\varepsilon'}(\Gamma_\phi), \qquad \text{for all } \Gamma' \in VG(\phi),$$

and then the theorem holds. Suppose from now on that $\varepsilon \in [0, v_2(\Gamma_\phi) - \underline{p}_2(\Gamma))$, which implies that

$$v(\Gamma_\phi) \neq \underline{p}(\Gamma). \tag{4.6}$$

Let us assume, for a moment, that the assertion in the theorem is not satisfied for Γ. This implies that there is an $\varepsilon_0 \in [0, v_2(\Gamma_\phi) - \underline{p}_2(\Gamma))$ and an $\varepsilon_1 > 0$, such that, for each $n \in \mathbb{N}$, there is a game $\Gamma^n = \langle X, Y, K_1^n, K_2^n\rangle \in VG(\phi)$ with $d(\Gamma^n,\Gamma) < \frac{1}{n}$, which possesses an $y^n \in O_2^{\varepsilon_0}(\Gamma_\phi^n)$ with $y^n \notin O_2^{\varepsilon_0+\varepsilon_1}(\Gamma_\phi)$. Then we have

$$\lim_{n\to\infty} d_H(\mathcal{R}(\Gamma^n), \mathcal{R}(\Gamma)) = 0, \tag{4.7}$$

and, for each $n \in \mathbb{N}$, there is an $x^n \in X$ with

$$\phi_2 K(x^n,y^n) < v_2(\Gamma_\phi) - \varepsilon_0 - \varepsilon_1, \tag{4.8}$$

$$\phi_2 K^n(x^n,y^n) \geq v_2(\Gamma_\phi^n) - \varepsilon_0. \tag{4.9}$$

Take a subsequence $n(1), n(2), \ldots$ of $1,2,\ldots$ such that the sequences $\langle K^{n(k)}(x^{n(k)},y^{n(k)});\ k \in \mathbb{N}\rangle$ and $\langle \phi_2 K^{n(k)}(x^{n(k)},y^{n(k)});\ k \in \mathbb{N}\rangle$ converge. Let $z = \lim_{k\to\infty} K^{n(k)}(x^{n(k)},y^{n(k)})$. By (4.7), $z \in \mathcal{R}(\Gamma)$. In view of (4.6) and Theorem 4.3 (iii), v_2 is continuous in Γ. The upper semi-continuity of ϕ_2, the continuity of v_2 in Γ and (4.7) and (4.9) imply that

$$(4.10)\qquad \begin{aligned}\phi_2(z) &\geq \lim_{k\to\infty} \phi_2 K^{n(k)}(x^{n(k)}, y^{n(k)}) \\ &\geq \liminf_{k\to\infty} v_2(\Gamma_\phi^{n(k)}) - \varepsilon_0 = v_2(\Gamma_\phi) - \varepsilon_0.\end{aligned}$$

It follows from (4.10) that $\phi_2(z) \geq v_2(\Gamma_\phi) - \varepsilon_0 > \underline{p}_2(\Gamma)$. Hence, ϕ_2 is continuous in $(z, \mathcal{R}(\Gamma))$, by Lemma 2.1. This implies, with (4.8) and the fact that $\lim_{k\to\infty} K(x^{n(k)}, y^{n(k)}) = z$, that

$$(4.11)\qquad \phi_2(z) = \lim_{k\to\infty} \phi_2 K(x^{n(k)}, y^{n(k)}) \leq v_2(\Gamma_\phi) - \varepsilon_0 - \varepsilon_1.$$

Since (4.10) and (4.11) contradict each other, we have completed the proof. □

5. CONTINUOUS ARBITRATION GAMES AND MIXED EXTENSIONS

In this section, we consider arbitration games, where the strategy spaces and the payoff functions of the underlying two-person game in normal form satisfy certain topological and algebraic assumptions. Furthermore, the underlying bargaining solution is u.s.c. and regular. We shall call a two-person game $\langle X,Y,K_1,K_2\rangle$ a *continuous game*, if

(i) X, Y are compact matric spaces,

(ii) K_1, K_2 are continuous functions on $X \times Y$.

The family of such continuous games will be denoted by $C(X,Y)$. For a $\Gamma \in C(X,Y)$, the game $\tilde{\Gamma} = \langle \tilde{X},\tilde{Y},\tilde{K}_1,\tilde{K}_2\rangle$ is called the *mixed extension* of Γ, where $\tilde{X}$ ($\tilde{Y}$) is the family of probability measures on the Borel sets of X (Y) and $\tilde{K}_i(\mu,\nu) = \int\int K_i(x,y)\,d\mu(x)\,d\nu(y)$, for $i \in \{1,2\}$ and $(\mu,\nu) \in \tilde{X}\times\tilde{Y}$. The family of mixed extensions of elements of $C(X,Y)$ is denoted by $MC(X,Y)$. We shall say that a game $\langle X,Y,K_1,K_2\rangle$ is a *convex continuous game* if

(i) X, Y are compact convex sets in topological vector spaces,

(ii) K_1, K_2 are continuous functions,

(iii) K_1 is concave in the first coordinate and convex in the second coordinate and K_2 is convex in the first coordinate and concave in the second coordinate.

The family of convex continuous games will be denoted by $CC(X,Y)$. Note that

$$(5.1) \qquad MC(Y,Y) \subset CC(\tilde{X},\tilde{Y}),$$

where $\tilde{X}$ and $\tilde{Y}$ are provided with the weak topology.

Now we want to prove a theorem, concerning the existence of equilibrium points. This result was proved for the first time in TIJS and JANSEN [10], with the help of the technique of dummy zero-sum games, developed in that paper. Now a new and simple proof will be given, using a theorem of BERGE [2], p.72.

We start with a

LEMMA 5.1. *Let* Γ_ϕ *be an arbitration game, where* ϕ *is a regular arbitration function. Let* $r,s \in R(\Gamma)$ *such that* $r_1 \le s_1$ *and* $r_2 \ge s_2$. *Then* $\phi_1(r) \le \phi_1(s)$.

PROOF. Suppose that $\phi_1(r) > \phi_1(s)$. Then $[r,\phi(r)] \cap [s,\phi(s)] \neq \emptyset$, which is impossible, in view of (A.3). □

THEOREM 5.2. *Let* Γ_ϕ *be an arbitration game, where* $\Gamma \in CC(X,Y)$ *and where* ϕ: $R(\Gamma) \to P(\Gamma)$ *is a regular arbitration function. Then* $E(\Gamma_\phi) \neq \emptyset$.

PROOF. First we show that $\phi_1 K$ is quasi-concave in the first coordinate i.e. $S(y,t) = \{x \in X;\ \phi_1 K(x,y) \ge t\}$ is a convex set for each $y \in Y$ and each $t \in \mathbb{R}$. Let $x^1,x^2 \in S(y,t)$ and $\alpha \in (0,1)$. Then, by (A.3),

$$(5.2) \qquad \phi_1(\alpha K(x^1,y) + (1-\alpha)K(x^2,y)) \ge t.$$

Since K_1 (K_2) is concave (convex) in the first coordinate, we have

$$K_1(\alpha x^1 + (1-\alpha)x^2,y) \ge \alpha K_1(x^1,y) + (1-\alpha)K_1(x^2,y)$$

$$K_2(\alpha x^1 + (1-\alpha)x^2,y) \le \alpha K_2(x^1,y) + (1-\alpha)K_2(x^2,y).$$

Application of Lemma 5.1, with $\alpha K(x^1,y) + (1-\alpha)K(x^2,y)$ in the role of r and $K(\alpha x^1 + (1-\alpha)x^2, y)$ in the role of s, yields, by (5.2):

$$\phi_1(K(\alpha x^1 + (1-\alpha)x^2,y)) \geq \phi_1(\alpha K(x^1,y) + (1-\alpha)K(x^2,y)) \geq t.$$

Hence, $\alpha x^1 + (1-\alpha)x^2 \in S(y,t)$. So $S(y,t)$ is convex. Similarly, one can prove that $\phi_2 K$ is quasi-concave in the second coordinate. Since $\phi_1 K$ and $\phi_1 K$ are also continuous, the equilibrium point theorem of Berge [2], p.72 can be applied, yielding $E(\Gamma_\phi) \neq \emptyset$. □

In view of (5.1), the following corollary is immediate.

COROLLARY 5.3. (Cf. [10], Theorem 5.7). *Let* $\tilde{\Gamma} \in MC(X,Y)$ *and let* ϕ: $R(\tilde{\Gamma}) \to P(\tilde{\Gamma})$ *be a regular arbitration function. Then* $E(\tilde{\Gamma}_\phi) \neq \emptyset$. □

Note that the existence results in NASH [8], RAIFFA [9] and KALAI and ROSENTHAL [5] are special cases of the above corollary.

We collect the important perturbation results for the class $CC(X,Y)$ in the following theorem.

THEOREM 5.4. *Let* ϕ: $\mathcal{B} \to \mathbb{R}^2$ *be an u.s.c. regular barginaing solution. Then*

(i) $CC(X,Y) \subset VG(\phi)$; $O_1(\Gamma_\phi)$ *and* $O_2(\Gamma_\phi)$ *are non-empty compact sets for each* $\Gamma \in CC(X,Y)$;

(ii) v_i: $CC(X,Y) \to \mathbb{R}$ *is, for each* $i \in \{1,2\}$, *an u.s.c. function;*

(iii) O_1: $CC(X,Y) \to X$, O_2: $CC(X,Y) \to Y$, *are u.s.c. multi-functions.*

PROOF.

(a) First we note that for an arbitration game Γ_ϕ we have $E(\Gamma_\phi) \neq \emptyset$ iff $v(\Gamma_\phi)$ exists and $O_1(\Gamma_\phi) \neq \emptyset$, $O_2(\Gamma_\phi) \neq \emptyset$. Hence, by Theorem 5.2 and Lemma 3.2, we have $CC(X,Y) \subset VG(\phi)$ and $O_i(\Gamma_\phi) \neq \emptyset$, for $i \in \{1,2\}$. That $O_i(\Gamma_\phi)$ is closed follows from the continuity of the map $\phi_i K$.

(b) The second assertion in the theorem follows from Theorem 4.3.

(c) Take a sequence $\Gamma, \Gamma^1, \Gamma^2, \ldots$ in $CC(X,Y)$ with $\lim_{n\to\infty} d(\Gamma^n,\Gamma) = 0$ and let $x, x^1, x^2, \ldots$ be a sequence in X with $x^n \in O_1(\Gamma^n_\phi)$, for each $n \in \mathbb{N}$, and $\lim_{n\to\infty} x^n = x$. Let $\delta > 0$. It follows from Theorem 4.5, with 0 and δ in the role of ε and ε', that $x^n \in O_1^\delta(\Gamma_\phi)$, for all n sufficiently large. But then $x \in O_1^\delta(\Gamma_\phi)$, since $O_1^\delta(\Gamma_\phi)$ is closed.

Since δ was arbitrarily chosen, $x \in \bigcap_{\delta>0} O_1^{\delta}(\Gamma_\phi) = O_1(\Gamma_\phi)$. This implies that O_1: $CC(X,Y) \to X$ is an u.s.c. multi-function. The second assertion in (iii) can be proved analogously. □

COROLLARY 5.5. *Let* ϕ: $B \to \mathbb{R}^2$ *be an u.s.c. regular bargaining solution. Then*
(i) v_i: $MC(X,Y) \to \mathbb{R}$ *is an u.s.c. function, for* $i \in \{1,2\}$.
(ii) O_1: $MC(X,Y) \to X$, O_2: $MC(X,Y) \to Y$ *are u.s.c. multi-functions.*

Now we consider mixed extensions of finite games. For $m,n \in \mathbb{N}$, let us denote the family of mixed extensions of m×n-bimatrix games by $F_{m,n}$. Let $A_{m,n}$ be the family of bargaining solutions ϕ, for which the value of the arbitration game Γ_ϕ exists for each $\Gamma \in F_{m,n}$.

Note that, in view of Corollary 5.3, each u.s.c. regular bargaining solution is an element of $A_{m,n}$. Let $C_{m,n}$ be the subset of $F_{m,n}$, consisting of elements Γ^* such that the map

$$\Gamma \to v_\phi(\Gamma) \qquad (\Gamma \in F_{m,n})$$

is continuous in Γ^*, for all $\phi \in A_{m,n}$.

THEOREM 5.6. *Let* $\Gamma \in F_{m,n}$. *The following assertions are equivalent.*
(1) $\Gamma \in C_{m,n}$;
(2) Γ *satisfies one of the four conditions in Theorem* 4.4.

PROOF. That (2) implies (1) follows immediately from Theorem 4.4. Suppose that $\Gamma' \in F_{m,n}$ does not satisfy (V.1) - (V.4). Then Γ' satisfies one of the following two conditions:

(H.1) $\qquad s_1(\Gamma') < \bar{p}_1(\Gamma')$ and $\bar{W}(\Gamma') \neq \{\bar{p}(\Gamma')\}$,

(H.2) $\qquad s_2(\Gamma') < \underline{p}_2(\Gamma')$ and $\underline{W}(\Gamma') \neq \underline{p}(\Gamma')$.

Suppose that (H.1) holds. We prove that $\Gamma \to v_{\psi^2}(\Gamma)$ is not continuous in Γ', where ψ^2 is the u.s.c. regular arbitration function, introduced in Section 2. Let $\bar{W}(\Gamma') = [c,\bar{p}(\Gamma')]$, and let Γ' be the mixed extension of the bimatrix game $[(a_{ij},b_{ij})]_{i=1}^{m}{}_{,}{}_{j=1}^{n}$. Then there is an $r \in \{1,\ldots m\}$

and an $s \in \{1,\ldots,n\}$, such that $c = (a_{rs}, b_{rs})$. For each $k \in \mathbb{N}$, let $[(a_{ij}, b_{ij}(k))]_{i=1}^{m}, {}_{j=1}^{n}$ be the bimatrix game with $b_{ij}(k) = b_{ij}$, if $i \neq r$ or $j \neq s$, and $b_{rs}(k) = b_{rs} + k^{-1}$ and denote the mixed extension of this game by Γ^k. Then $\lim_{k\to\infty} d(\Gamma^k, \Gamma') = 0$, $\lim_{k\to\infty} s_1(\Gamma^k) = s_1(\Gamma')$. But then $\lim_{k\to\infty} v_1(\Gamma^k_{\psi 2}) = \max\{c_1, s_1(\Gamma')\} \neq v_1(\Gamma'_{\psi 2}) = \bar{p}_1(\Gamma')$. Hence, $\Gamma' \notin C_{m,n}$. Similarly, if Γ' satisfies (H.2), then also $\Gamma' \notin C_{m,n}$. □

Let Γ be the mixed extension of the $m\times n$-bimatrix game $[(a_{ij}, b_{ij})]_{i=1}^{m}, {}_{j=1}^{n}$. We will say that $\Gamma \in \mathcal{D}_{m,n}$ if

(i) all entries a_{ij} in the payoff matrix A of player 1 are different.

(ii) all entries b_{ij} in the payoff matrix B of player 2 are different.

THEOREM 5.7.

(i) *$\mathcal{D}_{m,n}$ is an open and dense subset of $F_{m,n}$.*

(ii) *The value function $\Gamma \to v(\Gamma_\phi)$ is continuous on $\mathcal{D}_{m,n}$, for all $\phi \in A_{m,n}$.*

PROOF. (i) is obvious and (ii) follows from Theorem 4.4, because $P(\Gamma) = W(\Gamma)$, for each $\Gamma \in \mathcal{D}_{m,n}$. □

The following example shows that the multi-function $O_2 \colon MC(X,Y) \to Y$ is not necessarily lower semicontinuous.

EXAMPLE 5.8. Let $\Gamma \in F_{1,2}$ be the mixed extension of the bimatrix game $[(-3,0),(0,0)]$ and, for each $n \in \mathbb{N}$, let Γ^n be the mixed extension of the bimatrix game $[(-3,n^{-1}),(0,0)]$. Then $\lim_{n\to\infty} d(\Gamma^n,\Gamma) = 0$. For each bargaining solution ϕ we have: $O_2(\Gamma_\phi) = \{p \in \mathbb{R}^2;\ p \geq 0,\ p_1 + p_2 = 1\}$ and $O_2(\Gamma^n_\phi) = \{(1,0)\}$, for each $n \in \mathbb{N}$. So $\liminf_{n\to\infty} O_2(\Gamma^n_\phi) \not\supset O_2(\Gamma_\phi)$.

6. PERTURBATION OF BARGAINING SOLUTIONS

In this section, we consider a fixed game in normal form $\Gamma = \langle X,Y,K_1,K_2\rangle \in CC(X,Y)$ and vary the arbitration function. We are interested in the dependence of value and (ε-)optimal strategies on the arbitration function. Let Φ be the set of all regular u.s.c. bargaining solutions. To each $\phi \in \Phi$, there corresponds a regular arbitration function $\tilde{\phi} \colon R(\Gamma) \to P(\Gamma)$ in the obvious way (see Section 3). Let $\tilde{\Phi} = \{\tilde{\phi};\ \phi \in \Phi\}$. We provide $\tilde{\Phi}$ with a metric $\rho \colon \tilde{\Phi} \times \tilde{\Phi} \to \mathbb{R}$, defined by

$\rho(\tilde{\phi},\tilde{\psi}) = \sup_{r\in R(\Gamma)} \|\tilde{\phi}(r) - \tilde{\psi}(r)\|_\infty$, for all $\tilde{\phi},\tilde{\psi} \in \tilde{\Phi}$.

By Theorem 5.2, for each $\tilde{\phi} \in \tilde{\Phi}$, the value $v(\Gamma_{\tilde{\phi}})$ exists. The following continuity result holds.

THEOREM 6.1. *Let* $\varepsilon \geq 0$.

(i) $\tilde{\phi} \to v_i(\Gamma_{\tilde{\phi}})$ *is, for* $i \in \{1,2\}$, *a Lipschitz continuous function on* $\tilde{\Phi}$ *with Lipschitz constant* 1.

(ii) *If* $\rho(\tilde{\phi},\tilde{\psi}) \leq \delta$, *then* $O_i^\varepsilon(\Gamma_{\tilde{\psi}}) \subset O_i^{\varepsilon+2\delta}(\Gamma_{\tilde{\phi}})$, *for* $i \in \{1,2\}$.

PROOF. Obviously,

$$|v_1(\Gamma_{\tilde{\phi}}) - v_1(\Gamma_{\tilde{\psi}})|$$

$$= |\sup_{x\in X}\inf_{y\in Y} \tilde{\phi}_1 K(x,y) - \sup_{x\in X}\inf_{y\in Y} \tilde{\psi}_1 K(x,y)|$$

$$\leq \rho(\tilde{\phi},\tilde{\psi}).$$

Hence (i) holds. To prove (ii), let $\hat{x} \in O_1^\varepsilon(\Gamma_{\tilde{\psi}})$ and suppose that $\rho(\tilde{\phi},\tilde{\psi}) \leq \delta$. It follows from the inequalities

$$\tilde{\phi}_1 K(\hat{x},y) \geq \tilde{\psi}_1 K(\hat{x},y) - \delta, \qquad \text{for each } y \in Y,$$

$$\tilde{\psi}_1 K(\hat{x},y) \geq v_1(\Gamma_{\tilde{\psi}}) - \varepsilon, \qquad \text{for each } y \in Y,$$

$$v_1(\Gamma_{\tilde{\psi}}) \geq v_1(\Gamma_{\tilde{\psi}}) - \rho(\tilde{\phi},\tilde{\psi}) \geq v_1(\Gamma_{\tilde{\phi}}) - \delta,$$

that $\tilde{\phi}_1 K(\hat{x},y) \geq v_1(\Gamma_{\tilde{\phi}}) - \varepsilon - 2\delta$, for each $y \in Y$. Hence $\hat{x} \in O_1^{\varepsilon+2\delta}(\Gamma_{\tilde{\phi}})$. Thus $O_1^\varepsilon(\Gamma_{\tilde{\psi}}) \subset O_1^{\varepsilon+2\delta}(\Gamma_{\tilde{\phi}})$.

Similarly, one can prove the other inclusion in (ii). □

In AUMANN [1], p.547, the question was raised, under which circumstances optimal behaviour in arbitration games is independent of the arbitration function. An answer to this question is given in the following theorem. The answer, indicated in [1], seems not quite precise. First we note that the solution of the problem is trivial for games Γ with $\underline{p}(\Gamma) = \overline{p}(\Gamma)$, because then there is only one arbitration function and $E(\Gamma_\phi) = X \times Y$. In the following, we look at games with

$\underline{p}(\Gamma) \neq \bar{p}(\Gamma)$. For $z \in R(\Gamma)$, $z \notin \bar{W}(\Gamma) \cup \underline{W}(\Gamma)$, we define

$$f_1(z) = \begin{cases} (1,0) & \text{if } z_2 \geq \underline{p}_2(\Gamma) \\ \underline{p}(\Gamma) - z & \text{if } z_2 < \underline{p}_2(\Gamma) \end{cases},$$

$$f_2(z) = \begin{cases} (0,1) & \text{if } z_1 \geq \bar{p}_1(\Gamma) \\ \bar{p}(\Gamma) - z & \text{if } z_1 < \bar{p}_1(\Gamma). \end{cases}$$

Now, for $z \notin \bar{W}(\Gamma) \cup \underline{W}(\Gamma)$, $R(\Gamma)$ can be divided into four regions $R_1(z)$, $R_2(z)$, $R_3(z)$ and $R_4(z)$, where $R_i(z)$ consists of all elements in $R(\Gamma)$, which are of the form $r = z + \alpha_1 f_1(z) + \alpha_2 f_2(z)$, where $\alpha_1 \geq 0$, $\alpha_2 \geq 0$ if $i = 1$; $\alpha_1 \leq 0$, $\alpha_2 \geq 0$ if $i = 2$; $\alpha_1 \leq 0$, $\alpha_2 \leq 0$ if $i = 3$; $\alpha_1 \geq 0$, $\alpha_2 \leq 0$ if $i = 4$. Note that, for each regular ϕ, we have

$$(6.1) \qquad \phi_1(r) \leq \phi_1(z), \quad \text{for all } r \in R_2(z),$$

$$(6.2) \qquad \phi_2(r) \leq \phi_2(z), \quad \text{for all } r \in R_4(z).$$

In the proof of the subsequent theorem, we use the following arbitration functions. Let $\nu \in \mathbb{R}^2$, $\nu \geq 0$, $\nu_1 + \nu_2 > 0$ and let $r \in R(\Gamma)$. Let t be the unique point in $\{r + \alpha\nu \in R(\Gamma);\ \alpha \geq 0\}$ with largest second coordinate. Then define

$$\xi(\nu)(r) = \begin{cases} t & \text{if } t \in P(\Gamma) \\ \bar{p}(\Gamma) & \text{if } t_1 < \bar{p}_1(\Gamma) \\ \underline{p}(\Gamma) & \text{if } t_2 < \underline{p}_2(\Gamma). \end{cases}$$

Then $\xi(\nu)$ is a regular arbitration function.

<u>THEOREM 6.2</u>. *Let* $\Gamma = \langle X,Y,K_1,K_2\rangle$ *be a two-person game with* $\underline{p}(\Gamma) \neq \bar{p}(\Gamma)$. *Let* $(\hat{x},\hat{y}) \in X \times Y$ *and let* $z = K(\hat{x},\hat{y})$. *Suppose* $z \notin \bar{W}(\Gamma) \cup \underline{W}(\Gamma)$. *Then the following two assertions are equivalent.*

(i) $(\hat{x},\hat{y})$ *is an equilibrium point of* Γ_ϕ, *for each* $\phi \in \Phi$;

(ii) $K(x,\hat{y}) \in R_2(z)$, *for all* $x \in X$ *and* $K(\hat{x},y) \in R_4(z)$, *for all* $y \in Y$.

<u>PROOF</u>. Suppose that (ii) holds. Then, by (6.1) and (6.2), we have, for

all $\phi \in \Phi$,

$$\phi_1 K(x,\hat{y}) \le \phi_1 K(\hat{x},\hat{y}), \quad \text{for all } x \in X,$$

$$\phi_2 K(\hat{x},y) \le \phi_2 K(\hat{x},\hat{y}), \quad \text{for all } y \in Y.$$

Hence, $(\hat{x},\hat{y}) \in E(\Gamma_\phi)$, for each $\phi \in \Phi$. So (ii) implies (i). Now suppose that (ii) does not hold. Then there are two cases:

(1) there is an $x_0 \in X$ with $K(x_0,\hat{y}) \notin R_2(z)$;

(2) there is an $y_0 \in Y$ with $K(\hat{x},y_0) \notin R_4(z)$.

Suppose that we are in Case 1. If $K(x^0,\hat{y}) \in R_3(z)$, then $(\hat{x},\hat{y})$ is not an equilibrium point for the regular arbitration function $\xi(\upsilon^0)$, where $\upsilon^0 = f_1(z) + q - K(x^0,\hat{y})$ and where q is the unique point in $\{z + \alpha f_1(z) \in P(\Gamma);\ \alpha \ge 0\}$, because

$$\xi_1(\upsilon^0)(K(x_0,\hat{y})) > \xi_1(\upsilon^0)(K(\hat{x},\hat{y})).$$

If $K(x^0,\hat{y}) \notin R_3(z)$, then $(\hat{x},\hat{y})$ is not an equilibrium point for the regular arbitration function $\xi(\upsilon^1)$, with $\upsilon^1 = f_2(z)$, because $\xi_1(\upsilon^1)K(x_0,\hat{y}) > \xi_1(\upsilon^1)K(\hat{x},\hat{y})$. Hence, in Case 1, (i) does not hold. Similarly, the proof runs for Case 2. □

<u>REMARK 6.3</u>. If in Theorem 6.2, $z \in \overline{W}(\Gamma)$, then (i) holds iff

$$K(x,\hat{y}) \in \overline{W}(\Gamma), \quad \text{for all } x \in X.$$

If $z \in \underline{W}(\Gamma)$, then (i) holds iff

$$K(\hat{x},y) \in \underline{W}(\Gamma), \quad \text{for all } y \in Y.$$

In the following theorem, we give necessary and sufficient conditions, guaranteeing that the value of an arbitration game does not depend on the arbitration function.

<u>THEOREM 6.4</u>. *Let* $\Gamma \in CC(X,Y)$ *with* $\underline{p}(\Gamma) \ne \overline{p}(\Gamma)$. *Let* $z \in P(\Gamma) - \{\underline{p}(\Gamma), \overline{p}(\Gamma)\}$. *Then*

(i) $v(\Gamma_\phi) = \bar{p}(\Gamma)$ *for all* $\phi \in \Phi$ *iff there exists a* $\hat{y} \in Y$, *such that* $K_2(x,\hat{y}) = \bar{p}_2(\Gamma)$, *for all* $x \in X$.

(ii) $v(\Gamma_\phi) = \underline{p}(\Gamma)$ *for all* $\phi \in \Phi$ *iff there exists an* $\hat{x} \in X$, *such that* $K_1(\hat{x},y) = \underline{p}_1(\Gamma)$, *for all* $y \in Y$.

(iii) $v(\Gamma_\phi) = z$ *for all* $\phi \in \Phi$ *iff there exist* $\hat{x} \in X$ *and* $\hat{y} \in Y$ *such that* $K(\hat{x},\hat{y}) = z$ *and such that* $K_1(\hat{x},y) \geq z_1$, *for all* $y \in Y$ *and* $K_2(x,\hat{y}) \geq z_2$, *for all* $x \in X$.

PROOF.

(a) Suppose that $v(\Gamma_\phi) = \bar{p}(\Gamma)$ for all $\phi \in \Phi$. Take $\hat{y} \in O_2(\Gamma_{\psi^1})$. Then $\bar{p}_2(\Gamma) \leq \psi_2^1 K(x,\hat{y}) = K_2(x,\hat{y})$ for all $x \in X$. Conversely, suppose there is a $\hat{y} \in Y$, such that $K_2(x,\hat{y}) = \bar{p}_2(\Gamma)$. Then, for each $\phi \in \Phi$, $\bar{p}_2(\Gamma) \geq v_2(\Gamma_\phi) \geq \inf_x \phi_2 K(x,\hat{y}) = \bar{p}_2(\Gamma)$, which implies that $v(\Gamma_\phi) = \bar{p}(\Gamma)$. So (i) is proved.

Assertion (ii) can be proved in a similar way.

(b) Suppose $v(\Gamma_\phi) = z$, for all $\phi \in \Phi$. Take $\hat{x} \in O_1(\Gamma_{\psi^2})$ and $\hat{y} \in O_2(\Gamma_{\psi^1})$. Then $(\hat{x},\hat{y})$ has the properties, mentioned in (iii). Conversely, suppose that $(\hat{x},\hat{y}) \in X \times Y$ satisfies the properties mentioned in (iii). Then, for all $\phi \in \Phi$ and $x \in X$, $y \in Y$, we have

$$\phi_1 K(\hat{x},\hat{y}) = K_1(\hat{x},\hat{y}) \leq K_1(\hat{x},y) \leq \phi_1 K(\hat{x},y)$$

$$\phi_2 K(\hat{x},\hat{y}) = K_2(\hat{x},\hat{y}) \leq K_2(x,\hat{y}) \leq \phi_2 K(x,\hat{y}).$$

But then $v_1(\Gamma_\phi) \geq \inf_y \phi_1 K(\hat{x},y) = K_1(\hat{x},\hat{y}) = z_1$

$$v_2(\Gamma_\phi) \geq \inf_x \phi_2 K(x,\hat{y}) = K_2(\hat{x},\hat{y}) = z_2.$$

Hence, $v(\Gamma_\phi) \geq z$. Since $z \in P(\Gamma)$ and $v(\Gamma_\phi) \in P(\Gamma)$, we have $z = v(\Gamma_\phi)$. This completes the proof. □

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